PHYS 320 ANALYTICAL MECHANICS

Dr. Gregory W. Clark
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Calculus of Variations and Hamilton's Principle

$$J = \int_{x_1}^{x_2} f(y(x), y'(x); x)\,dx \quad \text{where} \quad y'(x) \equiv dy/dx$$

has extreme values (is "stationary") when

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Similarly

$$S = \int_{t_1}^{t_2} L\,dt \quad \text{where} \quad L \equiv T - U$$

Hamilton's principle

has extreme values (is stationary) when taken along the actual path, so that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

Lagrange's equation

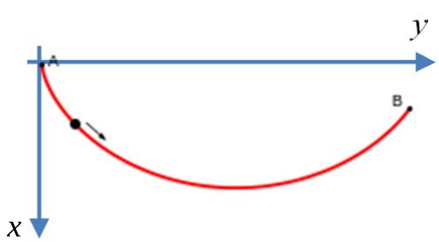
where the q_i are generalized coordinates

The Brachistochrone problem

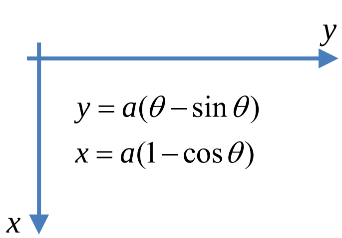
Johann Bernoulli (1696):
I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.

The problem: *Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.*

The Brachistochrone problem

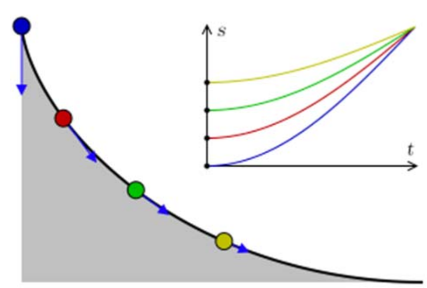


What constrained path allows a Particle to transit in the least possible Time from A to B in a constant force Field (*e.g.*, gravity) starting from rest?



$$y = a(\theta - \sin\theta)$$
$$x = a(1 - \cos\theta)$$

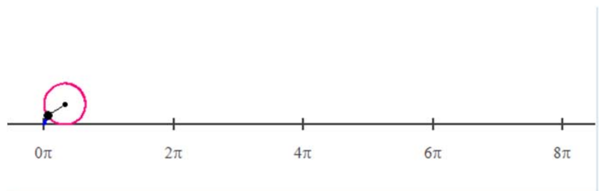
The cycloid path:



Recall: Rolling motion

- How describe …
  - the combination of circular and linear motion: rolling motion?

$$\vec{r} = \vec{r}_1 + \vec{r}_2 = \hat{i}\, b(\omega t + \sin \omega t) + \hat{j}\, b(1 + \cos \omega t)$$



Lagrangian Mechanics

Lagrangian function:

$$L \equiv T(q_j, \dot{q}_j, t) - U(q_j, t)$$

Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial q_j} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}, \quad j = 1, 2, 3, \ldots, s$$